A self-similar tiling generated by the minimal Pisot number

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Abstract

In [4] Thurston showed tilings of the plane by using Pisot numbers. In this paper we show a sufficient condition for two tiles to be adjacent in the case of the minimal Pisot number.

1 β-expansion

Let $\beta > 1$ be a real number. A *representation* in base β (or a β-representation) of a real number $x \geq 0$ is an infinite sequence $(x_i)_{k \geq i > -\infty}, x_i \geq 0$, such that

$$x = x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots x_1\beta + x_0 + x_{-1}\beta^{-1} + x_{-2}\beta^{-2} + \cdots$$

for a certain integer $k \geq 0$. It is denoted by

$$x = x_k x_{k-1} \cdots x_1 x_0 . x_{-1} x_{-2} \cdots$$

A particular β-representation – called the β-expansion – can be computed by the 'greedy algorithm': Denote by $[y]$ and $\{y\}$ the integer part and the fractional part of a number y. There exists $k \in \mathbf{Z}$ such that $\beta^k \leq x < \beta^{k+1}$. Let $x_k = [x/\beta^k]$, and $r_k = \{x/\beta^k\}$. Then for $k > i > -\infty$, put $x_i = [\beta r_{i+1}]$, and $r_i =$

$\{\beta r_{i+1}\}$. We get an expansion $x = x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots$. If $k < 0$ $(x < 1)$, we put $x_0 = x_{-1} = \cdots = x_{k+1} = 0$. If an expansion ends in infinitely many zeros, it is said to be *finite*, and the ending zeros are omitted.

The digits x_i obtained by this algorithm are integers from the set $\mathcal{A} = \{0, \ldots, \beta-1\}$ if β is an integer, or the set $\mathcal{A} = \{0, \ldots, [\beta]\}$ if β is not an integer.

A particular β-representation of 1, $d(1,\beta)$ – called the carry sequence of β – is defined by means of the *β-transformation* of the unit interval:

$$T_\beta x = \beta x (mod 1), \quad x \in [0,1].$$

$$d(1,\beta) = 0.t_{-1}t_{-2}\ldots, \quad t_{-k} = [\beta T_\beta^{k-1} 1],$$

Propositon 1. ([4]) *Let β be a real number greater than one, A β-representation of a number is the β-expansion if and only if the sequence of digits starting at any point is lexicographically less than the carry sequence $d(1,\beta)$.*

If a real number x has finite β-expansion, $x = x_k\beta^k + x_{k-1}\beta^{k-1} + \ldots x_t\beta^t$, $(x_k, x_t \neq 0)$, then we denote $deg_\beta(x) = k$ and $ord_\beta(x) = t$. $\mathbf{Fin}(\beta)$ is the set of numbers who have finite β-expansion. $\mathbf{Fin}_m(\beta)$ is the set of numbers whose ord_β is greater than m.

2 Statement of the result

Definition 1. *A* Pisot number *is an algebraic integer such that all its Galois conjugates are strictly inside the unit circle.*

In the following of this paper, we regard β is a complex Pisot number of degere three which is a unit; i.e. β is a real root of a irreducible polynomial of the following form:

$$x^3 - ax^2 - bx - 1, \quad a, b \in \mathbf{Z}$$

which has only one real root greater than 1. For any real number $a \in \mathbf{F}in(\beta)$, let $\mathbf{S}_a$ consist of all real numbers whose β-expansion agree with β-expansion of a after decimal point. Let α be a β's conjugate over $\mathbf{Q}$ other than β itself, and ϕ be the conjugate map over $\mathbf{Q}$ which transforms β to α. We denote $\phi(x)$ by x' and for any set $\mathbf{S} \subset \mathbf{Q}(\beta)$, $\mathbf{S}'$ denotes $\phi(\mathbf{S})$. It is clear that

$$\beta^{-1}\mathbf{S}_0 = \mathbf{S}_0 \cup \mathbf{S}_{0.1}, \quad \mathbf{S}_0 \cap \mathbf{S}_{0.1} = \emptyset.$$

Conjugating both side of the forms above

$$\alpha^{-1}\mathbf{S}'_0 = \mathbf{S}'_0 \cup \mathbf{S}'_{0.1}. \quad \mathbf{S}'_0 \cap \mathbf{S}'_{0.1} = \emptyset.$$

And in general,

$$\alpha^{-1}(\mathbf{Fin}_m(\beta))' = (\mathbf{Fin}_{m+1}(\beta))', \quad \mathbf{S}'_a \cap \mathbf{S}'_b = \emptyset \quad (\mathbf{S}_a \neq \mathbf{S}_b).$$

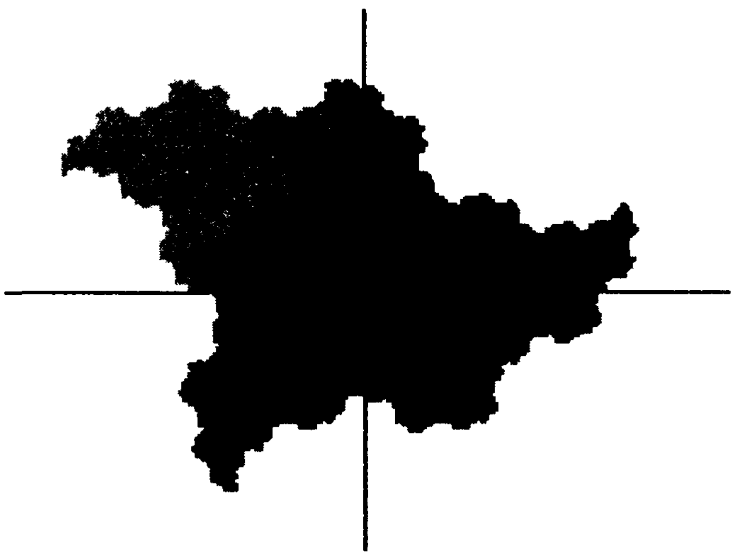

Figure 1: K_0, $K_{0.1}$ $: x^3 - x - 1$.

Let K_x be the closure of $\mathbf{S}'_x$ in $\mathbf{C}$. From figure 2 $\{K_x | x \in \mathbf{Fin}(\beta)\}$ seems to be a self-similar tiling of $\mathbf{C}$ with expansion constant α^{-1}. But threre is not any proof in [4]. In this paper we

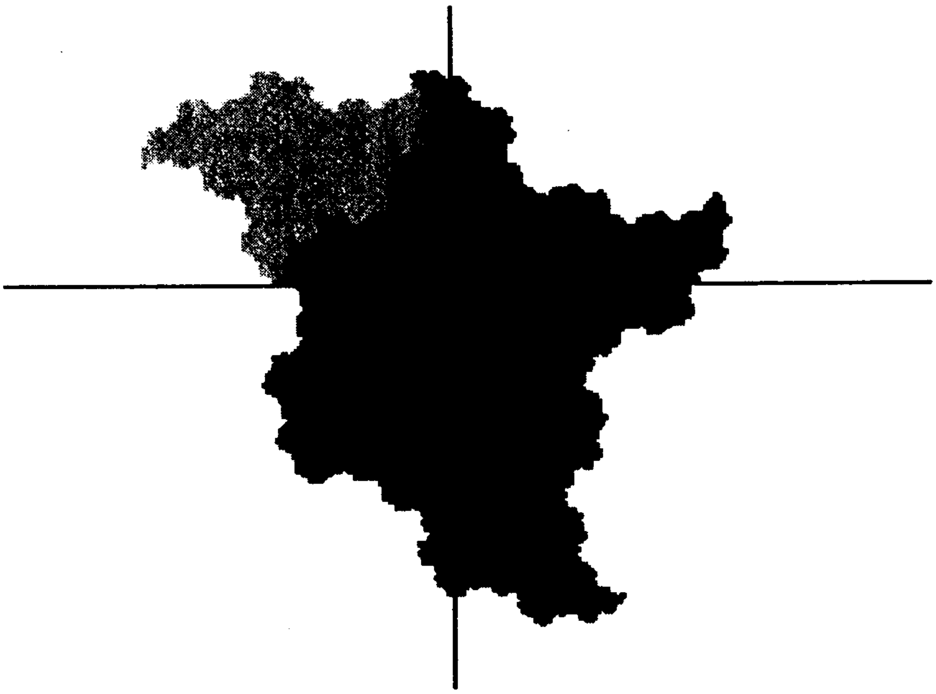

Figure 2: K_0, $K_{0.1}$, $K_{0.01}$: $x^3 - x - 1$.

prove some properties of $\{K_x | x \in \mathbf{Fin}(\beta)\}$ when β is the real root of $x^3 - x - 1$.

Propositon 2. *Let β be the real root of $x^3 - x - 1 = 0$. If $\mathbf{S}_a \neq \mathbf{S}_b$ then $\mu(K_a \cap K_b) = 0$, where μ denotes Lebesgue measure.*

Proof. $d(1, \beta) = 0.(10000)^\infty = 0.100001000010\cdots$. So from the proposition 1 if $x = x_k x_{k-1} \ldots$ be the β-expansion of x and $x_i = 1$ then $x_{i-1}, x_{i-2}, x_{i-3}, x_{i-4} = 0$.

$|\alpha|^{-2} = \beta$. It suffices to show when $a = 0$, $b = 0.1$. $\mathbf{S}_0 \cup \mathbf{S}_{0.1} = \beta^{-1}\mathbf{S}_0$, $\mathbf{S}_{0.1} = \beta^{-1} + \beta^{-4}\mathbf{S}_0$. Then $K_0 \cup K_{0.1} = \alpha^{-1}K_0$, $K_{0.1} = \alpha^{-1} + \alpha^4 K_0$.

$$\mu(K_0 \cup K_{0.1}) = |\alpha^{-1}|^2 \mu(K_0) = \beta\mu(K_0).$$

$$\mu(K_{0.1}) = |\alpha^4|^2 \mu(K_0) = \beta^{-4}\mu(K_0).$$

$$\begin{aligned} \mu(K_0 \cap K_{0.1}) &= \mu(K_0) + \mu(K_{0.1}) - \mu(K_0 \cup K_{0.1}) \\ &= (1 + \beta^{-4} - \beta)\mu(K_0) \\ &= 0. \end{aligned}$$

□

Theorem 1. *Let β be the real root of the polynomial $x^3 - x - 1$. Let a and b be nonnegative reals less than 1, and their β-expansions be finite. If the following conditions are satisfied, then $K_a \cap K_b$ is an infinite set.*

- *For an integer $d \geq -5$, $a - b = \beta^d$.*
- $\deg_\beta(a) \leq d + 4, \quad \deg_\beta(b) \leq d$

We prove the theorem by constructing the points on the intersection of two tiles.

3 Proof

Lemma 1. ([1], [2]) *Let β be the real root of the polynomial $x^3 - x - 1$. If $x \in \mathbf{Z}[\beta]$ then β-expansion of x is finite. So for any two nonnegative numbers x and y that have finite β-expansion, β-expansion of $|x \pm y|$ is also finite.*

Lemma 2. *Let β be the real root of the polynomial x^3-x-1, and $\{a_n\} \subset \mathbf{Z}[\beta]$ be a sequence of nonnegative numbers. If $\mathrm{ord}_\beta(a_n) \to \infty$ then $|a_n'| \to 0$.*

Proof. Suppose $h_n = \deg_\beta(a_n)$, $t_n = \mathrm{ord}_\beta(a_n)$.

$$a_n = c_{n,h_n}\beta^h + c_{n,h_n-1}\beta^{h_n-1} + \cdots + c_{n,t_n}\beta^{t_n}.$$

$$\begin{aligned} |a_n'| &\leq c_{n,h_n}|\alpha|^{h_n} + c_{n,h_n-1}|\alpha|^{h_n-1} + \cdots + c_{n,t_n}|\alpha|^{t_n} \\ &\leq [\beta]\frac{|\alpha|^{t_n}(1-|\alpha|^{h_n-t_n+1})}{1-|\alpha|} \\ &\leq [\beta]\frac{|\alpha|^{t_n}}{1-|\alpha|} = [\beta]\frac{|\alpha|^{ord_\beta(a_n)}}{1-|\alpha|}. \end{aligned}$$

□

Lemma 3. *$K_a \cap K_b \neq \emptyset$ if there exist two sequences $\{a_n\} \subset \mathbf{S}_a$, $\{b_n\} \subset \mathbf{S}_b$ such that $\{a_n'\}, \{b_n'\}$ converge in $\mathbf{C}$ and $\mathrm{ord}_\beta(|a_n - b_n|) \to \infty$.*

Proof. It follows from lemma 1 and 2. □

Note that the converse of the lemma 3 is also true, but it is not necessary for our purpose here.

Proof. (proof of the Theorem) We use the relations, $\beta^3 = \beta+1$ and $\beta^5 = \beta^4+1$. Let $a = 0.a_{-1}a_{-2}\cdots a_{-t}$, $b = 0.b_{-1}b_{-2}\cdots b_{-s}$ $t, s \in \mathbf{Z}$ be the β-expansions and $a - b = \beta^d$ for an integer d. We define sequences, $\{A_n\}$ and $\{B_n\}$ by the following procedure. $A_0 = a, B_0 = b$. When $A_n - B_n = \beta^{d_n}$, let $A_{n+1} = A_n$ $B_{n+1} = B_n +$

$\beta^{d_n+c_n}$ where $c_n = 5, or 3$. When $B_n - A_n = \beta^{d_n}$, let $A_{n+1} = A_n + \beta^{d_n+c_n}$ and $B_{n+1} = B_n$ where $c_n = 5, or 3$. If we choose $c_n = 5$, then

$$\begin{aligned} |A_{n+1} - B_{n+1}| &= \beta^{d_n+5} - |A_n - B_n| \\ &= \beta^{d_n+5} - \beta^{d_n} = \beta^{d_n}(\beta^5 - 1) \\ &= \beta^{d_n+4}, \end{aligned}$$

and hence

$$\text{ord}(|A_{n+1} - B_{n+1}|) = d_n + 4 = \text{ord}(|A_n - B_n|) + 4.$$

Similarly, if we choose $c_n = 3$, then

$$\text{ord}(|A_{n+1} - B_{n+1}|) = d_n + 1 = \text{ord}(|A_n - B_n|) + 1.$$

Repeating the procedure above, we can obtain sequences $\{A_n\}$ and $\{B_n\}$ such that $\text{ord}_\beta(|A_n - B_n|) \to \infty$. But the condition that $A_{n+1} \in \mathbf{S}_a$ and $B_{n+1} \in \mathbf{S}_b$ may not hold in the process. So we add the following restriction. Let $M_n = \deg_\beta(\max\{A_n, B_n\}) - d_n$, $m_n = \deg_\beta(\min\{A_n, B_n\}) - d_n$.

$$c_n = \begin{cases} 5 \ or \ 3 & m_n \le -2, d_n \ge -3 \\ 5 & m_n \le -2, \ -4 \ge d_n \ge -5 \\ 5 & -2 < m_n \le 0 \\ stop & other \end{cases}$$

Then $c_n - m_n \ge 5$, so β-expansions of A_{n+1} (resp. B_{n+1}) coinsides with A_n (resp. B_n) after decimal point. So $A_{n+1} \in \mathbf{S}_a$. If we choose $c_n = 5$, then $(M_{n+1}, m_{n+1}) = (1, M_n - 4)$, and if we choose $c_n = 3$, then $(M_{n+1}, m_{n+1}) = (2, M_n - 1)$. Let $A_0 = a$, $B_0 = b$, $A_1 = a$, $B_1 = b+\beta^{d+5}$, $A_2 = a+\beta^{d+9}$, $B_2 = b+\beta^{d+5}$, Then $(M_2, m_2) = (1, -3)$. For such A_2 and B_2, for $n > 2$ only m_n determines how many ways we can choose c_n and Figure 3 shows that infinitely many sequences can be obtained.

Let E_a be the set of all of the sequences that are obtained from the process above. Then E_a is an infinite set. We have to

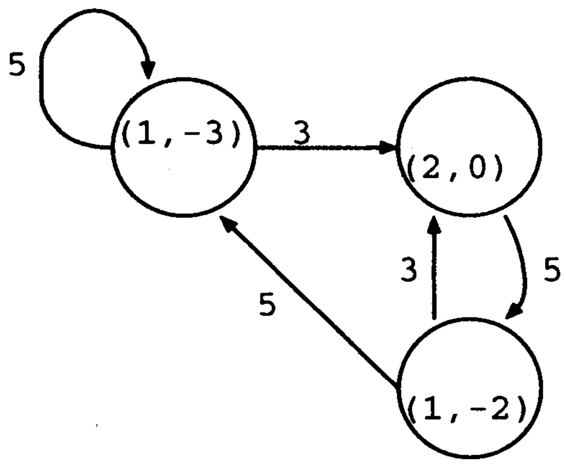

Figure 3: (M_n, m_n) and c_n

show the set of limit points of elements of E_a is also an infinite set. Suppose if n elements in E_a converge to the same point. We can construct a point that is included in n tiles So only finitely many sequences can converge to the same point, and hence the set of all of the limit points of elements of E_a is also an infinite set. □

Figure 4 shows a subset of $K_0 \cap (K_{0.1} \cup K_{0.01} \cup K_{0.001} \cup K_{0.0001} \cup K_{0.00001})$ obtained from the process above. It seems to be all of the intersection.

References

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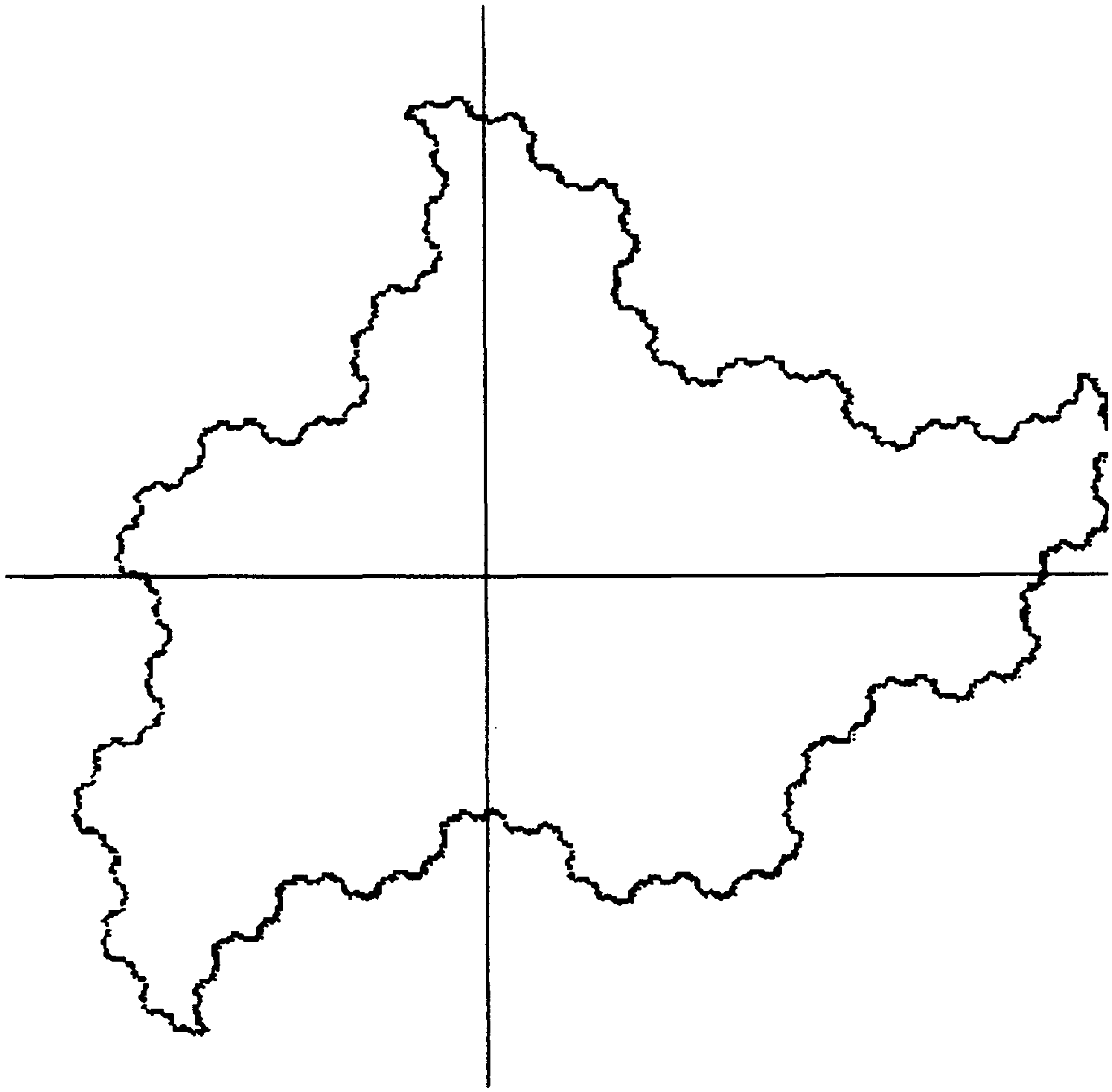

Figure 4: $K_0 \cap (K_{0.1} \cup K_{0.01} \cup K_{0.001} \cup K_{0.0001} \cup K_{0.00001})$.

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